MTH 305: Practice assignment 2

1 Division algorithm

Establish the following assertions using the division algorithm.

(i) The square of any integer is either of the form $3k$ or $3k+1$.

(ii) The cube of any integer is of the form $7k$ or $7k \pm 1$.

(iii) For $n \geq 1$, $\dfrac{n(n+1)(2n+1)}{6}$ is an integer.

(iv) The number $3n^2 - 1$, for $n \in \mathbb{Z}$, is never a perfect square.

(v) No integer in the sequence

$$11, 111, 1111, 11111, \ldots$$

is a perfect square.

(vi) When n is an odd integer, $n^4 + 4n^2 + 11$ is of the form $16k$.

2 Greatest common divisor

Establish the following assertions.

(i) $24 \mid (2 \cdot 7^n + 3 \cdot 5^n - 5)$, for $n \in \mathbb{Z}$.

(ii) If $a \in \mathbb{Z}$ is odd, then $32 \mid (a^2+3)(a^2+7)$

(iii) If $a, b \in \mathbb{Z}$ are odd, then $16 \mid (a^4 + b^4 - 2)$.

(iv) For any $a \in \mathbb{Z}$, $\gcd(a, a+1) = 1$.

(v) For any $a \in \mathbb{Z}$, $\gcd(5a+2, 7a+3) = 1$.

(vi) If $a \in \mathbb{Z}$ is odd, then $12 \mid (a^2 + (a+2)^2 + (a+4)^2 + 1)$.

(vii) If $a, b \in \mathbb{Z}$ are odd, then $8 \mid (a^2 - b^2)$.

(viii) If $a \in \mathbb{Z}$, then $360 \mid a^2(a^2-1)(a^2-4)$.

(ix) In each of the following assume that $\gcd(a, b) = 1$.

- (a) If $\gcd(a, c) = 1$, then $\gcd(a, bc) = 1$.
- (b) If $c \mid a$, then $\gcd(b, c) = 1$.
- (c) $\gcd(ac, b) = \gcd(c, b)$.
- (d) If $c \mid a + b$, then $\gcd(a, c) = \gcd(b, c) = 1$.
- (e) $\gcd(a^2, b^2) = 1$.

(x) If $d \mid n$, then $2^d - 1 \mid 2^n - 1$.

3 Euclid algorithm and Linear Diophantine equations

(i) In each of the following, find $\gcd(a, b)$ and express $\gcd(a, b)$ as a linear combination of a and b.

- (a) $(a, b) = (24, 138)$.
- (b) $(a, b) = (119, 272)$
- (c) $(a, b) = (306, 657)$

(ii) Establish the following assertions assuming that $\gcd(a, b) = 1$.

- (a) $\gcd(a+b, a-b) = 1$ or 2
- (b) $\gcd(a+b, a^2+b^2) = 1$ or 2
- (c) $\gcd(a+b, a^2-ab+b^2) = 1$ or 3
- (d) $\gcd(a^n, b^n) = 1$
- (e) $\gcd(a+b, ab) = 1$

(iii) For nonzero integers a and b, establish the following assertions.

(a) $\gcd(a,b) = \operatorname{lcm}(a,b) \iff a = \pm b$.

(b) $\operatorname{lcm}(ka, kb) = k \operatorname{lcm}(a,b)$.

(c) If $a \mid m$ and $b \mid m$, then $\operatorname{lcm}(a,b) \mid m$.

(iv) Let a, b, c be integers, no two of which are zero, and $d = \gcd(a,b,c)$.

(a) Show that

$$d = \gcd(\gcd(a,b), c) = \gcd(a, \gcd(b,c)) = \gcd(\gcd(a,c), b).$$

(b) Does a similar result as in (a) hold true if d is replaced with $\ell = \operatorname{lcm}(a,b,c)$ and gcd is replaced with lcm? Explain.

(v) Determine whether the following Diophantine equations can be solved. If they have solutions, determine all possible solutions in the integers.

(a) $33x + 14y = 115$.

(b) $24x + 138y = 18$.

(c) $54x + 21y = 906$.

(d) $123x + 360y = 99$.

(vi) If a and b are relatively prime positive integers, then show that Diophantine equation $ax - by = c$ has infinitely many solutions in the positive integers.